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On the Roman bondage number of graphs on surfaces

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Abstract A Roman dominating function on a graph G is a labeling $f : V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number, $\gamma_R(G)$, of G is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. The Roman bondage number $b_R(G)$ is the cardinality of a smallest set of edges whose removal from G results in a graph with Roman domination number greater than $\gamma_R(G)$. In this paper we obtain upper bounds on $b_R(G)$ in terms of (a) the average degree and maximum degree, and (b) Euler characteristic, girth and maximum degree. We also show that the Roman bondage number of every graph which admits a 2-cell embedding on a surface with non-negative Euler characteristic does not exceed 15.

Keywords Roman domination; Roman bondage number; girth; average degree; Euler characteristic
AMS subject classifications 05C69

1 Introduction

All graphs considered in this article are finite, undirected, without loops and multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. Let P_n denote the path with n vertices. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the *degree* of x is $d_G(x) = |N_G(x)|$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a graph G, let $x \in X \subseteq V(G)$. A vertex $y \in V(G)$ is an *X-private neighbor* of x if $N_G[y] \cap X = \{x\}$. The *X-private neighborhood* of x, denoted $pn_G[x, X]$, is the set of all X-private neighbors of x. An *orientable compact* 2*-manifold* $\mathbb{S}_h$ or *orientable surface* $\mathbb{S}_h$ *of genus* h (see [19]) is obtained from the sphere by adding h handles. Correspondingly, a *non-orientable compact* 2*-manifold* $\mathbb{N}_q$ or *non-orientable surface* $\mathbb{N}_q$ *of genus* q is obtained from the sphere by adding q crosscaps. Compact 2-manifolds are called simply surfaces throughout the paper. The *Euler characteristic* is defined by $\chi(\mathbb{S}_h) = 2 - 2h$, $h \geq 0$, and $\chi(\mathbb{N}_q) = 2 - q$, $q \geq 1$. The *Euclidean plane* $\mathbb{S}_0$, the *projective plane* $\mathbb{N}_1$, the *torus* $\mathbb{S}_1$, and the *Klein bottle* $\mathbb{N}_2$ are all the surfaces of non-negative Euler characteristic.

A *dominating set* for a graph G is a subset $D \subseteq V(G)$ of vertices such that every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. Graph domination applies naturally to many tasks, including facility location and network construction. For example, in constructing a cellular phone network, one needs to choose locations for the towers to cover a large region as cheaply as possible. Many variants of domination have been studied extensively and have applications such as constructions of error-correcting codes for digital communication and efficient data routing in wireless networks. A variation of domination called *Roman domination* was introduced independently by Arquilla and Fredricksen [2], ReVelle [16, 17] and Stewart [24], which was motivated with the following legend. In the 4th century A.D., Constantine the Great issued a decree to ensure the protection of the Roman empire. Constantine ordered that each city in the empire either has a legion stationed within it for defense or lies near a city with two standing legions. This way, if a defenseless city were attacked, a nearby city could dispatch reinforcements without leaving itself defenseless. The natural problem is to determine how few legions suffice to protect the empire. The concept of Roman domination can be formulated in terms of graphs. More formally, following Cockayne et al. [5], a *Roman dominating function* (or RDF) on a graph G is a vertex labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. For a RDF f, let $V_i^f = \{v \in V(G) : f(v) = i\}$ for each $i \in \{0, 1, 2\}$. Since this partition determines f, we can equivalently write $f = (V_0^f; V_1^f; V_2^f)$. The *weight* $f(V(G))$ of a RDF f on G is the value $\sum_{v \in V(G)} f(v)$, which equals $|V_1^f| + 2|V_2^f|$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A function $f = (V_0^f; V_1^f; V_2^f)$ is called a *γ_R-function* on G if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$. Cockayne et al. [5] showed that $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. Chambers et al. [4] proved that if a graph G has order n then $\gamma_R(G) \leq 4n/5$ when $\delta(G) \geq 1$ and $\gamma_R(G) \leq 8n/11$ when $\delta(G) \geq 2$. Liu and Chang [11] proved that $\gamma_R(G) \leq 2n/3$ when $\delta(G) \geq 3$. Liedloff et al. [10] and Liu and Chang [12] investigated algorithmic aspect of Roman domination. Applications of Roman domination were also shown in [4].

One measure of the stability of the Roman domination number of a graph G under edge removal is the *Roman bondage number* $b_R(G)$, defined by Rad and Volkmann in [13], as the cardinality of a smallest set of edges whose removal from G results in a graph with Roman domination number greater than $\gamma_R(G)$. In [3], Bahremandpour et al. showed that the decision problem for $b_R(G)$ is NP-hard even for bipartite graphs. For more information we refer the reader to [1, 3, 8, 13, 15, 21, 22, 25].

It is quite natural to consider the Roman bondage number for a graph on surfaces. The first upper bounds on $b_R(G)$, where G is a planar graph, were obtained by Rad and Volkmann [15]. They proved that $b_R(G) \leq \Delta(G) + 6$. In [1], Akbari, Khatirinejad and Qajar recently proved that $b_R(G) \leq 15$ provided G is a planar graph. In this paper we prove that 15 is an upper bound for $b_R(G)$ even when a graph G admits a 2-cell embedding on a surface $\mathbb{M} \in \{\mathbb{S}_1, \mathbb{N}_1, \mathbb{N}_2\}$. We also obtain upper bounds for $b_R(G)$ in terms of (a) average degree and maximum degree, and (b) Euler characteristic, girth and maximum degree.

2 Known results

The following results are important for our investigations.

Theorem A *Let G be a connected graph embeddable on a surface $\mathbb{M}$ whose Euler characteristic $\chi(\mathbb{M})$ is non-negative and let $\delta(G) \geq 5$. Then G contains an edge $e = xy$ with $d_G(x) + d_G(y) \leq 11$ if one of the following holds:*

(i) (Wernicke [26], Sanders [23]) $\mathbb{M} \in \{\mathbb{S}_0, \mathbb{N}_1\}$.

(ii) (Jendrol', Voss [9]) $\mathbb{M} \in \{\mathbb{S}_1, \mathbb{N}_2\}$ *and* $\Delta(G) \geq 7$.

Lemma B (Rad, Volkmann [14]) *If G is a graph, then $\gamma_R(G) \leq \gamma_R(G-e) \leq \gamma_R(G) + 1$ for any edge $e \in E(G)$.*

According to the effects of vertex removal on the Roman domination number of a graph G, let

- $V_R^+(G) = \{v \in V(G) \mid \gamma_R(G-v) > \gamma_R(G)\}$,
- $V_R^-(G) = \{v \in V(G) \mid \gamma_R(G-v) < \gamma_R(G)\}$,
- $V_R^0(G) = \{v \in V(G) \mid \gamma_R(G-v) = \gamma_R(G)\}$.

Clearly $\{V_R^-(G), V_R^0(G), V_R^+(G)\}$ is a partition of $V(G)$.

Theorem C (Rad, Volkmann [14]) *Let G be a graph of order at least 2.*

(i) If $v \in V_R^+(G)$ then for every γ_R-function $f = (V_0^f; V_1^f; V_2^f)$ on G, $f(v) = 2$ and $|pn_G[v, V_2^f] \cap V_0^f| \geq 3$.

(ii) For any vertex $u \in V(G)$, $\gamma_R(G) - 1 \leq \gamma_R(G - u)$.

Theorem D (Hansberg, Rad, Volkmann [6]) *Let v be a vertex of a graph G. Then $\gamma_R(G - v) < \gamma_R(G)$ if and only if there is a γ_R-function $f = (V_0^f, V_1^f, V_2^f)$ on G such that $v \in V_1^f$.*

Theorem E (Rad, Volkmann [13]) *If G is a graph, and xyz is a path of length 2 in G, then*

$$b_R(G) \leq d_G(x) + d_G(y) + d_G(z) - 3 - |N_G(x) \cap N_G(y)|.$$

The *average degree* $ad(G)$ of a graph G is defined as $ad(G) = 2|E(G)|/|V(G)|$.

Theorem F (Hartnell, Rall [7]) *For any connected graph G of order at least two, there exists a pair of vertices, say u and v, that are either adjacent or at distance 2 from each other, with the property that $d_G(u) + d_G(v) \leq 2ad(G)$.*

The *girth* of a graph G is the length of a shortest cycle in G; the girth of a forest is ∞.

Lemma G (Samodivkin [22]) *Let G be a connected graph embeddable on a surface $\mathbb{M}$ whose Euler characteristic χ is as large as possible and let the girth of G is $k < \infty$. Then:*

$$ad(G) \leq \frac{2k}{k-2}\left(1 - \frac{\chi}{|V(G)|}\right).$$

Given a graph G of order n, let $\widehat{G}$ be the graph of order $5n$ obtained from G by identifying the central vertex of a copy of P_5, to each vertex of G.

Lemma H (Akbari, Khatirinejad, Qajar [1]) *Let G be a graph of order n, $n \geq 2$. Then $\gamma(\widehat{G}) = 2n$, $\gamma_R(\widehat{G}) = 4n$ and $b_R(\widehat{G}) = \delta(G) + 2$.*

3 Upper bounds

A graph G of order at least two is *Roman domination vertex critical* if removing any vertex of G decreases the Roman domination number. By $\mathcal{R}_{CV}$ we denote the class of all Roman domination vertex critical graphs. Results on this class can be found in Rad and Volkmann [14] and Hansberg et al. [6].

Theorem 1 *Let G be a connected graph.*

(i) If $V_R^-(G) \neq V(G)$ then

$$b_R(G) \leq \min\{d_G(u) - \gamma_R(G-u) + \gamma_R(G) \mid u \in V_R^0(G) \cup V_R^+(G)\} \leq \Delta(G).$$

(ii) If $b_R(G) > \Delta(G)$ then G is in $\mathcal{R}_{CV}$.

A leaf of a graph is a vertex of degree 1, while a support vertex is a vertex adjacent to a leaf.

Remark 2 *Let G be any connected graph of order $n \geq 2$. Let S be the set consisting of all support vertices of $\widehat{G}$. Then $f = (V(\widehat{G}) - S; \emptyset; S)$ is a RDF on $\widehat{G}$. Since the weight of f is $4n$, by Lemma H it follows that f is a γ_R-function on $\widehat{G}$. Theorem C(i) now implies $V(\widehat{G}) = V_R^-(\widehat{G}) \cup V_R^0(\widehat{G})$. Since $\gamma_R(P_5) = 4$ and since the central vertex of P_5 is in $V_R^0(P_5)$, $V(G) \subset V_R^0(\widehat{G})$. Labeling the vertices of each P_5 of $\widehat{G}$ with $(1,1,0,2,0)$ yields a γ_R-function on $\widehat{G}$. It follows by Theorem D that $V(G) = V_R^0(\widehat{G})$. All this together with Lemma H shows that the bound in Theorem 1(i) is attainable for all graphs $\widehat{G}$. Furthermore, for any graph $\widehat{G}$ the bound in Theorem E is attainable too. Indeed, let us consider any path xyz in $\widehat{G}$, where x is a leaf and $d_G(z) = \delta(G)$. Applying Theorem E to the path xyz we obtain $b_R(\widehat{G}) \leq d_{\widehat{G}}(z)$. Since $d_{\widehat{G}}(z) = \delta(G) + 2$, the result now follows by Lemma H.*

To prove Theorem 1, we need the following lemma:

Lemma 3 *Let G be a connected graph. For any subset $U \subsetneq V(G)$, let E_U denote the set of all edges between U and $V(G) - U$.*

(i) If $v \in V_R^0(G) \cup V_R^+(G)$ then $\gamma_R(G - E_{\{v\}}) > \gamma_R(G)$.

(ii) If $x \in V_R^+(G)$ then $1 \leq \gamma_R(G-x) - \gamma_R(G) \leq d_G(x) - 2$ and for any subset $S \subseteq E_{\{x\}}$ with $|S| \geq d_G(x) - \gamma_R(G-x) + \gamma_R(G)$, $\gamma_R(G-S) > \gamma_R(G)$.

Proof. (i) We have $\gamma_R(G - E_{\{v\}}) \geq \gamma_R(G-v) + 1 > \gamma_R(G)$.

(ii) Denote $p = \gamma_R(G-x) - \gamma_R(G)$. Let f be any γ_R-function on G. Since $p > 0$, by Theorem C(i) it follows that $f(x) = 2$. Hence $h = (V_0^f - N_G(x); V_1^f \cup (N_G(x) - V_2^f); V_2^f - \{x\})$ is a RDF on $G - x$. But then $\gamma_R(G) + p = \gamma_R(G-x) \leq h(V(G-x)) \leq \gamma_R(G) + d_G(x) - 2$. Hence $1 \leq p \leq d_G(x) - 2$. For any set $S \subseteq E_{\{x\}}$ with $|S| \geq d_G(x) - p$ we have $\gamma_R(G-S) \geq \gamma_R(G - E_{\{x\}}) - (|E_{\{x\}}| - |S|) \geq (\gamma_R(G-x)+1) - d_G(x) + (d_G(x) - p) =$

$\gamma_R(G)+1$, where the first inequality follows from Lemma B. □

Proof of Theorem 1. (i) The result follows immediately by Lemma 3.
(ii) Immediately by (i). □

Rad and Volkmann [15] as well as Akbari et al. [1] gave upper bounds on the Roman bondage number of planar graphs. Upper bounds on the Roman bondage number of graphs 2-cell embeddable on topological surfaces in terms of orientable/non-orientable genus and maximum degree, are obtained by the present author in [21].

Theorem 4 *Let G be a connected graph with* $\Delta(G) \geq 2$.

(i) Then $b_R(G) \leq 2ad(G) + \Delta(G) - 3$.

(ii) Let G be embeddable on a surface $\mathbb{M}$ *whose Euler characteristic* χ *is as large as possible. If G has order n and girth* $k < \infty$ *then:*

$$b_R(G) \leq \frac{4k}{k-2}\left(1 - \frac{\chi}{n}\right) + \Delta(G) - 3.$$

Proof. (i) If G is a complete graph then the result is obvious. Hence we may assume G has non-adjacent vertices. Theorem F implies that there are 2 vertices, say x and y, that are either adjacent or at distance 2 from each other, with the property that $d_G(x)+d_G(y) \leq 2ad(G)$. Since G is connected and $\Delta(G) \geq 2$, there is a vertex z such that xyz or xzy is a path. It follows from Theorem E that $b_R(G) \leq d_G(x)+d_G(y)+d_G(z)-3 \leq 2ad(G)+\Delta(G)-3$.
(ii) Lemma G and (i) together imply the result. □

Remark 5 *Let* $\mathbb{M}$ *be a surface. Denote* $\delta^{\mathbb{M}}_{\max} = \max\{\delta(H) \mid H$ is a graph 2-cell embedded in $\mathbb{M}\}$. *Let G be a connected graph* 2*-cell embeddable on* $\mathbb{M}$ *and* $\delta(G) = \delta^{\mathbb{M}}_{\max}$. *By Lemma H it immediately follows* $b_R(\widehat{G}) = \delta^{\mathbb{M}}_{\max} + 2$. *Note that (a) if* $\chi(\mathbb{M}) \leq 1$ *then* $\delta^{\mathbb{M}}_{\max} \leq \lfloor (5+\sqrt{49-24\chi(\mathbb{M})})/2 \rfloor$ *(see Sachs [20], pp. 226–227), and (b) it is well known that* $\delta^{\mathbb{S}_0}_{\max} = \delta^{\mathbb{N}_1}_{\max} = 5$ *and* $\delta^{\mathbb{N}_2}_{\max} = \delta^{\mathbb{S}_1}_{\max} = 6$.

In [1], Akbari, Khatirinejad and Qajar recently proved that $b_R(G) \leq 15$ provided G is a planar graph. As the next result shows, more is true.

Theorem 6 *Let G be a connected graph* 2*-cell embedded on a surface* $\mathbb{M}$ *with non-negative Euler characteristic and let* $\Delta(G) \geq 2$. *Then* $b_R(G) \leq 15$.

Proof. If $2 \leq \Delta(G) \leq 6$ then $b_R(G) \leq 3\Delta(G) - 3 \leq 15$, because of Theorem E. So, assume $\Delta(G) \geq 7$. Denote $V_{\leq 5} = \{v \in V(G) \mid d_G(v) \leq 5\}$ and $G_{\geq 6} = G - V_{\leq 5}$. Since $\chi(\mathbb{M}) \geq 0$,

$\delta(G) \leq 6$ (see Remark 5). If $\delta(G) = 6$ then G is a 6-regular triangulation on the torus or in the Klein Bottle, a contradiction with $\Delta(G) \geq 7$. So, $\delta(G) \leq 5$ and then $V_{\leq 5}$ is not empty. Since $G_{\geq 6}$ is embedded without crossings on $\mathbb{M}$ and $\chi(\mathbb{M}) \geq 0$, there is a vertex $u \in V(G_{\geq 6})$ with $d_{G_{\geq 6}}(u) \leq 6$. If u has exactly 2 neighbors belonging to $V_{\leq 5}$ then again by Theorem E, $b_R(G) \leq 15$. Now let all $v_1, v_2, v_3 \in V_{\leq 5}$ be adjacent to u. Denote by E_1 the set of all edges of G which are incident to at least one of v_1, v_2 and v_3. Since v_1, v_2 and v_3 are isolated vertices in $G - E_1$, for any γ_R-function g on $G - E_1$, $g(v_1) = g(v_2) = g(v_3) = 1$. Define now $f : V(G) \to \{0,1,2\}$ by $f(v_1) = f(v_2) = f(v_3) = 0$, $f(u) = 2$ and $f(v) = g(v)$ for every $v \in V(G) - \{u, v_1, v_2, v_3\}$. Clearly f is a RDF on G with $\gamma_R(G) \leq f(V(G)) < g(V(G - E_1)) = \gamma_R(G - E_1)$. Thus, $b_R(G) \leq |E_1| \leq d_G(v_1) + d_G(v_2) + d_G(v_3) \leq 15$.

So, it remains to consider the case where each vertex of degree at most 6 in $G_{\geq 6}$ has no more than one neighbor in $V_{\leq 5}$. It immediately follows that $\delta(G_{\geq 6}) \geq 5$. First assume $\delta(G_{\geq 6}) = 5$. By Theorem A it follows that there is an edge $xy \in E(G_{\geq 6})$ such that $d_{G_{\geq 6}}(x) + d_{G_{\geq 6}}(y) \leq 11$. Hence $d_G(x) + d_G(y) \leq 13$. Let without loss of generality $d_{G_{\geq 6}}(x) \leq d_{G_{\geq 6}}(y)$. Then x has exactly one neighbor in $V_{\leq 5}$, say v. By Theorem E applied to the path v, x, y we have $b_R(G) \leq 5 + 13 - 3 = 15$. Now let $\delta(G_{\geq 6}) \geq 6$. But then $G_{\geq 6}$ is a 6-regular triangulation on the torus or in the Klein bottle. Since $\Delta(G) \geq 7$, $G \neq G_{\geq 6}$ and there is a path x, y, z in G, where $d_G(z) \leq 5$, and both x and y are in $V(G_{\geq 6})$. Since clearly $|N(x) \cap N(y)| \geq 2$, again using Theorem E we obtain $b_R(G) \leq 7 + 7 + 5 - 3 - 2 = 14$. $\square$

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